A study of the fat realization of a bar construction

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We show that for a bisimplicial manifold $NG(*) \rtimes NH(*)$, a fat realization of a bar construction $\overline{W}_*(NG \rtimes NH)$ is homeomorphic to a fat realization of a diagonal simplicial manifold $d(N(G \rtimes H))(*)$.

1 Introduction

For a bisimplicial set $X_{*,*}$, there are two ways to extract from it a simplicial set and it is known that the geometric realizations of these simplicial sets are homotopy equivalent ([1] [3]). On the other hand, for any simplicial set a space called its fat realization is also constructed. In this paper, we show that for a bisimplicial manifold $NG(*) \rtimes NH(*)$, a fat realization of a bar construction $\overline{W}_*(NG \rtimes NH)$ is homeomorphic to a fat realization of a diagonal simplicial manifold $d(N(G \rtimes H))(*)$.

2 A bisimplicial set

A bisimplicial set is a sequence of sets with horizontal face maps ε_i^{Ho}, vertical face maps ε_i^{Ve}, horizontal degeneracy maps η_i^{Ho} and vertical degeneracy maps η_i^{Ve} which commute with each other.

For any bisimplicial set $\{X_{*,*}\}$, we can associate a topological space $|X_{*,*}|$ called the geometric realization defined as follows:

$$|X_{*,*}| := \coprod_{n,m} \Delta^n \times \Delta^m \times X_{n,m} / \sim .$$

Here Δ^n is the standard n-simplex and the equivalent relation $\sim$ is defined as:

$$((\varepsilon^i \times \mathrm{id})(t,s),x) \sim ((t,s),\varepsilon_i^{Ho}x),$$

$$((\mathrm{id} \times \varepsilon^i)(t,s),x) \sim ((t,s),\varepsilon_i^{Ve}x),$$

$$((\eta^i \times \mathrm{id})(t,s),x) \sim ((t,s),\eta_i^{Ho}x),$$

$$((\mathrm{id} \times \eta^i)(t,s),x) \sim ((t,s),\eta_i^{Ve}x).$$

3 The constructions of dX and $\overline{\mathrm{W}}X$

In this section we recall the definitions and properties of dX and $\overline{\mathrm{W}}X$.

Definition 3.1. For a bisimplicial set $X_{*,*}$, a simplicial set $(dX)_*$ is defined as $(dX)_p := X_{p,p}$.

Definition 3.2. For a bisimplicial set $X_{*,*}$, a simplicial set $\overline{\mathrm{W}}_*X$ is defined as follows:

$$\overline{\mathrm{W}}_pX := \{(x_0,x_1,\cdots,x_p) \in \prod_{i=0}^{p} X_{i,p-i} |$$

$$x_i \in X_{i,p-i}, \varepsilon_0^{Ve}x_i = \varepsilon_{i+1}^{Ho}x_{i+1}, \text{for all } 0 \leq i < p\}.$$

Face operators are given by

$$\varepsilon_i^{\overline{W}}(x_0,\cdots,x_p) =$$

$$(\varepsilon_i^{Ve}x_0, \varepsilon_{i-1}^{Ve}x_1, \cdots, \varepsilon_1^{Ve}x_{i-1},$$

$$\varepsilon_i^{Ho}x_{i+1}, \varepsilon_i^{Ho}x_{i+2}, \cdots, \varepsilon_i^{Ho}x_p).$$

Theorem 3.1 (D.Quillen [2]). *For any bisimplicial set $X_{*,*}$, the geometric realization $|X_{*,*}|$ and $|(dX)_*|$ are homotopic.*

Theorem 3.2 (A.Cegarra and J.Remedios[1], D. Stevenson[3]). *For any bisimplicial set $X_{*,*}$, the geometric realization $|(dX)_*|$ and $|\overline{\mathrm{W}}_* X|$ are homotopic.*

4 A bisimplicial manifold $NG(*) \rtimes NH(*)$

Let G be a Lie group and H be a subgroup of G. We define a bisimplicial manifold $NG(*) \rtimes NH(*)$ as follows:

$$NG(p) \rtimes NH(q) := \overbrace{G \times \cdots \times G}^{p-times} \times \overbrace{H \times \cdots \times H}^{q-times}.$$

Horizontal face operators $\varepsilon_i^{Ho} : NG(p) \rtimes NH(q) \to NG(p-1) \rtimes NH(q)$ are the same as the face operators of $NG(p)$. Vertical face operators $\varepsilon_i^{Ve} : NG(p) \rtimes NH(q) \to NG(p) \rtimes NH(q-1)$ are

$$\varepsilon_i^{Ve}(\vec{g}, h_1, \cdots, h_q) = \begin{cases} (\vec{g}, h_2, \cdots, h_q) & i = 0 \\ (\vec{g}, h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_q \vec{g} h_q^{-1}, h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

Here $\vec{g} = (g_1, \cdots, g_p)$.

5 The fat realization

For any simplicial set X_*, a topological space $\| X_* \|$ called the fat realization is defined as follows:

$$\| X_* \| := \coprod_n \Delta^n \times X_n / \sim .$$

Here the equivalent relation $\sim$ is defined as $(\varepsilon^i t, x) \sim (t, \varepsilon_i x)$.

Proposition 5.1. *$\| d(N(G \rtimes H))(*) \|$ is homeomorphic to $\| \overline{W}_*(NG \rtimes NH) \|$.*

Proof. In this case any element $(x_0, x_1, \cdots, x_p) \in \overline{W}_p(NG \rtimes NH)$ is determined by $x_0 = (h_1, \cdots, h_p) \in NH(p)$ and $x_p = (g_1, \cdots, g_p) \in NG(p)$. Actually, $x_i \in NG(i) \rtimes NH(p-i)$ can be written as $x_i = (g_1, g_2, \cdots, g_i, h_{i+1}, \cdots, h_p)$. So there is a trivial homeomorphism $i : \overline{W}_p(NG \rtimes NH) \to NG(p) \rtimes NH(p)$ and this commutes with the face maps. □

References

[1] A. Cegarra and J.Remedios, The relationship between the diagonal and the bar constructions on a bisimplicial set, Topology Appl. 153 (2005), no.1, 21-51.

[2] D. Quillen, Higher algebraic K-theory: I, in: Algebraic K-Theory I, in: Lecture Notes in Math., vol. 341, Springer, Berlin, 1973, pp. 77-139.

[3] D. Stevenson, Décalage and Kan's simplicial loop group functor, Theory Appl. Categ. 26 (2012), 768-787.