On some characteristic classes of the universal torus bundle in the BSS complex

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We exhibit cocycles in the Bott-Shulman-Stasheff complex which represent the Chern characters and Chern classes of the universal torus bundle $ET^n \to BT^n$.

1. Introduction

For any Lie group G, there is a simplicial Lie group NG called nerve of G. We can construct a double complex $\Omega^*(NG(*))$ on NG and it is well-known that the cohomology ring of its total complex is isomorphic to $H^*(BG)$.

In [8], the author exhibited a cocycle in $\Omega^*(NG(*))$ which represents the Chern character in the case of $G = GL(n, \mathbb{C})$. Unfortunately, this cocycle is complicated and it will be more complicated if we try to exhibit a cocycle which represents the Chern class of $EGL(n, \mathbb{C}) \to BGL(n, \mathbb{C})$.

In this paper, we exhibit cocycles which represent the Chern character and the Chern class of the universal torus bundle $ET^n \to BT^n$ in $\Omega^*(NT^n(*))$. In this case, the cocycles can be written as much more simple form.

2. Review of the universal Chern-Weil Theory

In this section we recall the universal Chern-Weil theory following [4]. For any Lie group G, we have simplicial manifolds NG, PG and simplicial G-bundle $\gamma : PG \to NG$ defined as follows:

$$NG(q) = \overbrace{G \times \cdots \times G}^{q-times} \ni (h_1, \cdots, h_q):$$

face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(h_1, \cdots, h_q) =$$

$$\begin{cases} (h_2, \cdots, h_q) & i = 0 \\ (h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

$$PG(q) = \overbrace{G \times \cdots \times G}^{q+1-times} \ni (g_1, \cdots, g_{q+1}):$$

face operators $\bar{\varepsilon}_i : PG(q) \to PG(q-1)$

$$\bar{\varepsilon}_i(g_0, \cdots, g_q) = (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_q).$$

We define $\gamma : PG \to NG$ as $\gamma(g_0, \cdots, g_q) = (g_0 g_1^{-1}, \cdots, g_{q-1} g_q^{-1})$.

For any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\|X\|$ called the fat realization and it is well-known that $\|\gamma\|$ is a model of the universal bundle $EG \to BG$ [6].

Now we construct a double complex associated to a simplicial manifold.

Definition 2.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define a double complex as follows:

$$\Omega^{p,q}(X) := \Omega^q(X_p)$$

Derivatives are:

$$d' = \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*$$

$d'' = (-1)^p$ the exterior differential on $\Omega^*(X_p)$.

In this paper, we call the double complex $\Omega^{p,q}(NG)$ the BSS (Bott-Shulman-Stasheff) complex.

For NG and PG the following theorem holds [2] [4] [5].

Theorem 2.1. *There exist ring isomorphisms*

$$H(\Omega^*(NG)) \cong H^*(BG),$$

$$H(\Omega^*(PG)) \cong H^*(EG).$$

Here $\Omega^*(NG)$ *and* $\Omega^*(PG)$ *mean the total complexes.*

There is another double complex associated to a simplicial manifold.

Definition 2.2 ([3]). A simplicial n-form on a simplicial manifold $\{X_p\}$ is a sequence $\{\phi^{(p)}\}$ of n-forms $\phi^{(p)}$ on $\Delta^p \times X_p$ such that

$$(\varepsilon^i \times id)^* \phi^{(p)} = (id \times \varepsilon_i)^* \phi^{(p-1)}.$$

Here ε^i is the canonical i-th face operator of Δ^p.

Let $A^{k,l}(X)$ be the set of all simplicial $(k+l)$-forms on $\Delta^p \times X_p$ which are expressed locally of the form

$$\sum a_{i1} \cdots {}_{ikj1} \cdots {}_{jl} (dt_{i1} \wedge \cdots dt_{ik} \wedge dx_{j1} \wedge \cdots dx_{jl})$$

where $(t_0, t_1, \cdots, t_p)$ are the barycentric coordinates in Δ^p and x_j are the local coordinates in X_p. We define derivatives as:

d' = the exterior differential on Δ^p,

$d'' = (-1)^k \times$ the exterior differential on X_p.

Then $(A^{k,l}(X), d', d'')$ is a double complex.

Let $A^*(X)$ denote the total complex of $A^{*,*}(X)$. We define a map $I_\Delta : A^*(X) \to \Omega^*(X)$ as follows:

$$I_\Delta(\alpha) := \int_{\Delta p} (\alpha|_{\Delta p \times Xp}).$$

Then the following theorem holds [3].

Theorem 2.2. I_Δ *induces a natural ring isomorphism*

$$I_\Delta^* : H(A^*(X)) \cong H(\Omega^*(X)).$$

Let $\mathcal{G}$ denote the Lie algebra of G. A connection on a simplicial G-bundle $\pi : \{E_p\} \to \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $\mathcal{G}$ such that θ restricted to $\Delta^p \times E_p$ is a usual connection form on $\Delta^p \times E_p \to \Delta^p \times M_p$.

There is a canonical connection $\theta \in A^1(PG)$ on $\gamma : PG \to NG$ defined as follows:

$$\theta|_{\Delta p \times PG(p)} := t_0\theta_0 + \cdots + t_p\theta_p.$$

Here θ_i is defined as $\theta_i = \mathrm{pr}_i^* \bar{\theta}$ where $\mathrm{pr}_i : \Delta^p \times PG(p) \to G$ is the projection into the i-th factor of $PG(p)$ and $\bar{\theta}$ is the Maurer-Cartan form of G. We also obtain its curvature $\Omega \in A^2(PG)$ on γ as $\Omega|_{\Delta p \times PG(p)} = d\theta|_{\Delta p \times PG(p)} + \frac{1}{2}[\theta|_{\Delta p \times PG(p)}, \theta|_{\Delta p \times PG(p)}]$.

Let $(\mathrm{Sym}\, \mathcal{G}^*)^G$ denote the ring of G-invariant polynomials on $\mathcal{G}$. For $P \in (\mathrm{Sym}\, \mathcal{G}^*)^G$, we restrict $P(\Omega) \in A^*(PG)$ to each $\Delta^p \times PG(p) \to \Delta^p \times NG(p)$ and apply the usual Chern-Weil theory then we obtain a simplicial $2k$-form $P(\Omega)$ on NG.

Now we have a homomorphism

$$w : (\mathrm{Sym}\, \mathcal{G}^*)^G \to H(\Omega^*(NG))$$

which maps $P \in (\mathrm{Sym}\, \mathcal{G}^*)^G$ to $w(P) = [I_\Delta(P(\Omega))] \in H(\Omega^*(NG)) \cong H^*(BG)$.

3. The Chern charater in the BSS complex

In this section we take G as a n-dimensional torus T^n and exhibit a cocycle in $\Omega^*(NT^n)$ which represents the Chern character of the universal T^n-bundle $ET^n \to BT^n$.

We first give a cocycle in $\Omega^{p+q}(PT^n(p-q))$ $(0 \leq q \leq p-1)$ which corresponds to the p-th Chern character by restricting $(1/p!)\ \mathrm{tr}\,((-\Omega/2\pi i)^p) \in A^{2p}(PG)$ to $A^{p-q,p+q}(\Delta^{p-q} \times PG(p-q))$ and integrating it along Δ^{p-q}. Then we give a cocycle in $\Omega^{p+q}(NT^n(p-q))$ which is mapped to the cocycle in $\Omega^*(PT^n)$ by γ^*.

If we write the i-th factor $g_i \in PT^n(p)$ as a diagonal matrix $(w_k^i)_{1 \leq k \leq n}$, θ_i can be written as $\theta_i = ((w_k^i)^{-1} dw_k^i)_{1 \leq k \leq n}$ and we can see $[\theta_i, \theta_j] = 0$ for any i, j hence $\Omega|_{\Delta p-q \times PTn(p-q)} = -\sum_{i=1}^{p-q} dt_i \wedge (\theta_0 - \theta_i)$.

Now $dt_i \wedge (\theta_0 - \theta_i) = dt_i \wedge \{(\theta_0 - \theta_1) + (\theta_1 - \theta_2) + \cdots + (\theta_{i-1} - \theta_i)\}$ and for any $\mathcal{G}$-valued differential forms α, β, γ and any integer $0 \leq \forall x \leq p-q-1$, the equation $\alpha \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \gamma = -\alpha \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \gamma$ holds, so the terms of these forms cancel with each other in $(-\Omega|_{\Delta p-q \times PG(p-q)})^p$. Therefore we see $(-\Omega|_{\Delta p-q \times PG(p-q)})^p = (\sum_{i=1}^{p-q} dt_i \wedge (\theta_{i-1} - \theta_i))^p$ and this cochain vanishes unless $q = 0$.

As a consequence of this argument, we obtain the following theorem.

Theorem 3.1. *The cocycle* $\bar{\omega}_p$ in $\Omega^p(PT^n(p))$ *which corresponds to the p-th Chern character is given as follows*:

$$\bar{\omega}_p = (-1)^{p(p-1)/2} \frac{1}{p!} \left(\frac{1}{2\pi i}\right)^p \times \sum_{k=1}^{n} \left(\prod_{i=0}^{p-1} \left(\frac{dw_k^i}{w_k^i} - \frac{dw_k^{i+1}}{w_k^{i+1}} \right) \right).$$

We write the i-th factor of $NT^n(p-q)$ as a diagonal matrix $(z_k^i)_{1 \le k \le n}$. Then we can easily check that the equation

$$\gamma^* \left(\frac{dz_k^{i+1}}{z_k^{i+1}} \right) = \frac{dw_k^i}{w_k^i} - \frac{dw_k^{i+1}}{w_k^{i+1}}$$

holds true.

Now we are ready to state the main theorem.

Theorem 3.2. *The cocycle* ω_p *in* $\Omega^p(NT^n(p))$ *which represents to the p-th Chern character is given as follows*:

$$\omega_p = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left(\frac{1}{2\pi i}\right)^p \times \sum_{k=1}^{n} \left(\frac{dz_k^1}{z_k^1} \wedge \frac{dz_k^2}{z_k^2} \wedge \cdots \wedge \frac{dz_k^p}{z_k^p} \right).$$

Proof. The cocycle in Theorem 3.2 is mapped to the cocycle in Theorem 3.1 by $\gamma^* : \Omega^p(NT^n(p)) \to \Omega^p(PT^n(p))$. The statement folllows from this.

4. The Chern class in the BSS complex

By repeating the same argument in section 3, we obtain the cocycle in μ_p which represents the Chern class.

Theorem 4.1. *The cocycle* μ_p *in* $\Omega^p(NT^n(p))$ *which represents to the p-th Chern class is given as follows*:

$$\mu_p = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left(\frac{1}{2\pi i}\right)^p \times \sum_{1 \le k < \cdots < kp \le n} \det \left(\frac{dz_{kj}^i}{z_{kj}^i} \right)_{1 \le i,j \le p}.$$

Remark 4.1. In the case that n is equal to p, the cocycle in theorem 4.1 is written as:

$$\mu_p = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left(\frac{1}{2\pi i}\right)^p \det \left(\frac{dz_k^i}{z_k^i} \right)_{1 \le i,k \le p}.$$

If we write $z_k^i = \exp(i\theta_k^i)$, μ_p is equal to

$$(-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left(\frac{1}{2\pi}\right)^p \det (d\theta_k^i)_{1 \le i,k \le p}.$$

This cocycle represents the Euler class of the universal $SO(2)^p$-bundle in the BSS complex. For example, the following cocycle represents the Euler class of the universal $(SO(2) \times SO(2))$-bundle:

$$\frac{-1}{2} \left(\frac{1}{2\pi}\right)^2 (d\theta_1^1 d\theta_2^2 - d\theta_1^2 d\theta_2^1).$$

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