On Improving an Approximate Solution of a Nonlinear Two Point Boundary Value Problem by Deferred Corrections

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1. Basic Results and Definitions

Let us consider the nonlinear functional equation

$$F(v)=0 \tag{1}$$

where the continuous operator F maps a linear subspace D of a Banach space E into a banach space E^0. We will always assume that (1) has a unique solution $u \in D \subset E$.

We are interested in the approximate solution of (1) by means of discretization algorithms. A discretization of (1) consists of families depending on a real parameter $h \in H=(0, h_0]$, of

(a) Banach spaces E_h and E^0_h ;

(b) linear mappings $\Delta_h : E \rightarrow E_h$, $\Delta^0_h : E^0 \rightarrow E^0_h$

(c) functions (generally nonlinear) $\Phi_h : E_h \rightarrow E^0_h$

(d) argorithms

$$\Phi_h(V)=0. \tag{2}$$

The operators F, Φ_h will be assumed to have the following properties :

For each $v \in D$ and $h \in H$ there exists an expansion

$$\Phi_h(\Delta_h v)=\Delta^0_h\{F(v)+\sum_{j=1}^{N} h^{p_j} F_{p_j}(v)\}+0(h^{\bar{p}}) \tag{3}$$

where the operators F_{p_j} : $D \rightarrow E^0$ are given independently of h, $\bar{p}>p_N$. The exponents appearing in (3) will be positive rational numbers satisfying

$$0<p_1<p_2<\cdots<p_N.$$

The operators F and Φ_h will always be assumed to be at least twice Frèchet-differentiable on E and E_h.

Let q be a positive number. We will say that $U(h) \in E_h$ is an approximate solution of (2) if it satisfies

$$\|\Phi_h(U(h))\| \leqq Ch^q$$

where C is a positive constant.

If $h \in H$, u is the solution of (1), and U(h) is an approximate solution of (2), then the vector

$$e(h)=U(h)-\Delta_h u \in E_h$$

will be called the global discretization error of (2).

Let u be the unique solution of (1). Property

$$\|U(h)-\Delta_h u\|\to 0 \text{ as } h\to 0$$

will be called the discrete convergence of the approximate solution U(h). More precisely the method (2) having an asymptotic expansion (3) will be convergent of order p if for any $h\in H$, $\|e(h)\|< Ch^p$ where C is a positive constant. In this case U(h) will also be called a p-approximate solution of (2).

If for any $e\in E_h$, a fixed $V\in E_h$ and any $h\in H$ there exists a nonnegative constant K (which may be depend on V) such that

$$\|e\|\leqq K\|\Phi_h'(V)e\|,$$

then we will say that the operator $\Phi_h(V)$ is stable at V. This is equivalent to say that if $\Phi_h'(V)$ is onto, then it has an inverse and $\|(\Phi_h'(V))^{-1}\|\leqq K$.

A differentiable operator $\Phi_h(V)$ is said to have the mean value property if for each $V_1, V_2\in E_h$, there exists a linear operator $M(V_1, V_2)$ such that

$$\Phi_h(V_1)-\Phi_h(V_2)=M(V_1,V_2)(V_1-V_2)\ , \qquad (4)$$

and

$$\|M(V_1,V_2)-\Phi_h'(V)\|=o(1)$$

for $V_1, V_2\to V$.

Theorem 1. *Let u be the solution of (1) and U(h) an approximate solution of (2) with exponent $q\geqq p_2$. If an expansion (3) with $N=2$ is valid, Φ_h has the mean value property and the $M(U,\Delta_h u)$ of (4) has an inverse bounded in norm, then the method (2) is convergent of order p_1.*

We say that global discretization error has an asymptotic expansion up to the order $p_N>0$ if there exists $e_j\in E$, independent of $h\in H$, such that

$$\|e(h)-\Delta_h\sum_{j=1}^{N} h^{p_j} e_j\|\leqq C_N h^{\bar{p}}$$

where $C_N>0$ constant, and $\bar{p}>p_N$.

Theorem 2. *Let U(h) be a p_1-approximate solution with $q\geqq p_2$, and let u be the exact solution of (1). Let Φ_h and Φ_h' have an asymptotic expantion (3) up to the order p_2, with F_{p_1} independent of h. If Φ_h is stable at $\Delta_h u$, then the global error e(h) has an asymptotic expansion up to the order p_1, i.e.*

$$\|e(h)-\Delta_h h^{p_1} e_1\|\leqq Ch^{\bar{p}},\ \bar{p}>p_1 \qquad (5)$$

where e_1 is indepeendeut of h and satisfies

$$F'(u)e_1=-F_{p_1}u\ .$$

Once the expantion (5) has been secured, a deferred correction procedures are available in order to obtain a more accurate approximation than U(h).

Theorem 3. *Under the hypotheses of Theorem 2, and if there exists operator S_{p_1} such that*

$$\Delta_h^0 F_{p_1} u-S_{p_1}(U)=O(h^{p*})\ , \qquad (6)$$

where $p^* = \min(p_1, p_2 - np_1)$, *the*

$$U_1 = U - h^{p_1} e^* \tag{7}$$

is an approximate solution of (1) *of order* $\bar{p}$, *that is*

$$\|U_1 - \Delta_h u\| = O(h^{\bar{p}}).$$

Here e^* *is the solution of the linear problem*

$$\Phi'_h(U) e^* = -S_{p_1}(U). \tag{8}$$

For the proof of Theorems 1, 2, 3, see [3].

Remark. Since the exact solution of (2) be an approximation to the solution of (1), it is no use to solve (2) exactly. The condition $q \geqq p_1$ says how incomplete this approximate solution can be.

2. Application to Non-linear Two-point Boundary Value Problem

We want to solve the problem

$$y'' = f(x, y, y'); \qquad y(a) = 0, \qquad y(b) = 0, \tag{9}$$

where $f(x) \in C^{\infty}([a, b] \times R^2)$.

We will assume that $f(x, y, y')$ is sufficiently differentiable as a function of its three arguments, which in particular will imply that the solution of (9) has continuous derivatives up to the order necessary in the following discussion. In order to insure existence and uniqueness of a solution of (9) we will assume that

$$f_y(x, y, z) \geqq 0, \qquad |f_z(x, y, z)| < K$$

in a certain bounded region $\Omega = [a, b] \times B \times B'$. Let us call that solution $y(x)$.

Let us take D the Banach space of twice continuously differentiable functions on $[a, b]$ which satisfy homogeneous boundary conditions. The operator $F(y) = y'' - f(x, y, y')$ will map D into $E^0 = C[a, b]$. Let us subdivide the interval $[a, b]$ into n equal parts by defining $x_i = a + ih$; $h = (b-a)/n$. Let E_h be the $(n-1)$-dimensional linear space of $(n+1)$-component vectors V with $V_0 = V_n = 0$, and let $E_h^o = R_{n-1}$. The norms involved will be the L_∞-norms for vectors and matrices

Now we define for every $v \in E$, $w \in E^0$ the discretization mappings

$$\Delta_h v = \{v(x_i)\}_{i=0,1,\ldots,n}, \qquad \Delta_h^0 w = \{w(x_i)\}_{i=1,\ldots,n-1}.$$

We can now define a discrete version of (9) :

$$[\Phi_h(Y)]_j = h^{-2}(-Y_{j-1} + 2Y_j - Y_{j+1}) + f(x_j, Y_j, (Y_{j+1} - Y_{j-1})/2h) = 0, \quad (j = 1, \ldots, n-1) \tag{10}$$

which is defined for every $Y \in E_h$. The Frèchet derivative of Φ_h is

$$\{\Phi'_h(Y) e\}_j = h^{-2}\{-(1 + \frac{h}{2} f_z^j(Y)) e_{j-1} + (2 + h^2 f_y^j(Y) e_j - (1 - \frac{h}{2} f_z^j(Y)) e_{j+1}\}, \quad (j = 1, \ldots, n-1) \tag{11}$$

the notation for the partial derivatives of f in (11) is $f_z^j(Y) = f_z(x_j, Y_j, (Y_{j+1} - Y_{j-1})/2h)$, and so on. For any $v \in C^{\infty}[a, b] \subset E$ we have the following asymptotic expansion

$$\Phi_h(\Delta_h v) = \Delta_h^0 \left\{ F(v) + \sum_{j=1}^{N} h^{2j} \left[\frac{-2}{(2j+2)!} v^{(2j+2)} + g_{2j} \right] \right\} + O(h^{2N+1}), \tag{12}$$

where the functions g_{2j} can be obtained by reordering

$$\sum_{\nu=1}^{N}\frac{f_z^{(\nu)}(v)}{\nu!}\sum_{j=1}^{N}\left[\frac{h^{2j}}{(2j+1)!}v^{(2j+1)}\right]^{\nu}=h^2\left(f_z\frac{v^{(3)}}{3!}\right)+h^4\left(f_z\frac{v^{(5)}}{5!}+f_{zz}\frac{1}{2}\left(\frac{v^{(3)}}{3!}\right)^2\right)+h^6\cdots.$$

Since $\Phi_h(Y)$ clearly has the mean value property, if we are able to show that it is also stable then, by Theorem 1, we will have that it is convergent of order 2.

Now, in order to carry out the following discussion we introduce some definitions (see [1]).

Let W be the set of the first n integers, $W=\{1,2,\ldots,n\}$. A matrix $A=(a_{ij})$ is called reducible if it is possible to decompose W into two nonempty, disjoint subsets S and T, such that $a_{ij}=0$ for $i\in S$ and $j\in T$. A matrix which is not reducible is called irreducible.

Lemma 1 *A tridiagonal matrix $A=(a_{ij})$ is irreducible if and only if*

$$a_{i,i-1}\neq 0\ (i=2,3,\ldots,n)\quad\text{and}\quad a_{i,i+1}\neq 0\ (i=1,2,\ldots,n-1)$$

A matrix A with real elements is called monotone if $A_z\geq 0$ implies $z\geq 0$. A monotone matrix is nonsingular.

Lemma 2 *Let the matrix $A=(a_{ij})$ be irreducible and satisfy the conditions*

$$(i)\quad a_{ij}\leq 0,\quad i\neq j;\quad i,j=1,\ldots,n$$

$$(ii)\quad \sum_{j=1}^{N}a_{ij}\begin{cases}\geq 0, & \text{for } i=1,2,\ldots,n\\ >0, & \text{for at least one } i.\end{cases}$$

Then A is monotone.

From lemmas 1 and 2 it is clear that for sufficiently small h the operator $\Phi'_h(Y)$ of (11) is of monotonic type for any $Y\in E_h$. In order to obtain the inverse of the tridiagonal matrix $\Phi'_h(Y)$, we can use the L-U decomposition method[1], and from its procedure it is clear that

$$\|(\Phi'_h(Y))^{-1}\|\leq K,$$

and hence $\Phi_h(Y)$ is stable.

In conclusion Φ_h is stable and convergent of order 2, and we can apply the deferred correction argorithms.

Now for the linear, one-step correction, which will give a fourth order approximate solution, we will develope some special formulas in order to approximate $y'''(x)$ and $y^{(4)}(x)$ at the interior points.

Let U be an approximate solution of (10) with $q\geqq 2$, and define

$$\delta f_j=\delta f(x_j,U_j,(U_{j+1}-U_{j-1})/2h)$$

$$=f(x_{j+1},U_{j+1},(U_{j+2}-U_j)/2h)-f(x_{j-1},U_{j-1},(U_j-U_{j-2})/2h).$$

Since $e_j=U_j-y_j=0(h^2)$ we have

$$f\left(x,U,\frac{\delta U}{2h}\right)=f\left(x,e+y,\frac{\delta(e+y)}{2h}\right)=f(x,y,y')+f_ye+f_{y'}\left(\frac{\delta e}{2h}+\frac{y'''}{3!}\right)+0(h^4).$$

Moreover, since $y''(x)=f(x,y(x),y'(x))$ we have that

$$\delta_x y''(x)=\delta_x f(x,y(x),y'(x))=2hy'''(x)+0(h^3).$$

On the other hand, since $e_j=h^2e(x_j)+0(h^4)$, we have

$$\delta[f(x,U,\delta U/2h)-f(x,y,y')]=h^2\delta(f_ye(x)+f_{y'}(\delta e(x)/2h+y'''(x)/6))+0(h^4)$$

and

$$\delta[f(x)-f_j]/2h=h^2\frac{d}{dx}[f_y\cdot e(x)+f_{y'}\cdot(e'(x)+y'''(x))]+0(h^4)$$

and from the differentiability properties of all the involved functions we have that

$$\delta f(x)-\delta f_j=0(h^3).$$

Thus δf_j approaches $\delta_x f(x)$ as $0(h^3)$, and hence

$$y'''(x_j)-(\delta f_j)/2h=0(h^2).$$

Similarly if we define

$$\delta^2 f_j=f(x_{j+1},U_{j+1},(U_{j+2}-U_j)/2h)-2f(x_j,U_j,(U_{j+1}-U_{j-1})/2h)+f(x_{j-1},U_{j-1},(U_j-U_{j-2})/2h)$$

then $\quad y^{(4)}(x_j)-(\delta^2 f_j)/h^2=0(h^2).$

From these we can now define

$$\left[\ S_2(U)\ \right]_j=\frac{h^{-2}}{12}\delta^2 f_j-\frac{h^{-1}}{12}f^j_{y}{}'(U)\delta f_j$$

which satiafies condition (6) since also $f^j_y{}'(U)-f_{y'}(y(x_j))=0(h^2)$. Hence, by Theorem 3 we can obtain solving (8) and using (7) an approximate solution U_1 of order 4.

3. Numerical Example

Let us consider the two point boundary value problem

$$-u''=1+0.49(u')^2 \qquad u(0)=u(1)=0$$

whose solution is $u(x)=\frac{1}{0.49}\ln\left[\frac{\cos 0.7(x-1/2)}{\cos 0.7/2.}\right]$.

Using the deferred correction method in section 2, we could obtain the more accurate values than the values which we obtained using the quasilinearization method that discussed in [6], [7].

x	Quasilinearization Method	Deferred Correction Method	Exact Values	Errors by D.C.M.
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.046532	0.046578	0.046571	0.000007
0.2	0.082235	0.082316	0.082304	0.000012
0.3	0.107483	0.107587	0.107573	0.000014
0.4	0.122532	0.122649	0.122635	0.000014
0.5	0.127532	0.127653	0.127639	0.000014
0.6	0.122532	0.122649	0.122635	0.000014
0.7	0.107483	0.107587	0.107573	0.000014
0.8	0.082235	0.082316	0.082304	0.000012
0.9	0.046532	0.046578	0.046571	0.000007
1.0	0.000000	0.000000	0.000000	0.000000

References

[1] Henrici,P. : Discrete variable methods in ordinary differential equations. New York : Wiley 1962.

[2] Stetter,H.: Asymptotic expansions for the error of discretization algorithms for non-linear functional equations. Numer. Math. 7, 18—31 (1965) .

〔3〕 Pereyra, V. : On improving an approximate solution of a functional equation by deferred corrections. Numer. Math. 8, 376—391 (1966) .

〔4〕 Pereyra, V. : Iterated deferred corrections for nonlinear operator equations. Numer. Math. 10, 316—323 (1967) .

〔5〕 Pereyra, V. : Iterated deferred corrections for nonlinear boundary value problems. Numer. Math. 11, 111—125 (1968) .

〔6〕 Kudo, M. & Yoshimura, T. : On quadratic convergence of successive iterative method in nonlinear two-point boundary value problem. Akita Kōgyō Kōtō Senmon Gakkō Kiyō, 5, 104—108 (1969) .

〔7〕 Yoshimura, T. & Kudo, M. : On monotone convergence in quasilinearization method. Akita Kōgyō Kōtō Senmon Gakkō Kiyō, 5, 109—113 (1969) .

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